

NOTE ON LINEARLY EQUIVALENT IDEAL TOPOLOGIES OVER NOETHERIAN MODULES

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ABSTRACT. Let R be a commutative Noetherian ring, and let N be a non-zero finitely generated R -module. In this paper, the main result asserts that for any N -proper ideal \mathfrak{a} of R , the \mathfrak{a} -symbolic topology on N is linearly equivalent to the \mathfrak{a} -adic topology on N if and only if, for every $\mathfrak{p} \in \text{Supp}(N)$, $\text{Ass}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ consists of a single prime ideal and $\dim N \leq 1$.

1. INTRODUCTION

Let R be a commutative Noetherian ring, \mathfrak{a} an ideal of R and let N be a non-zero finitely generated R -module. For a non-negative integer n , the n th *symbolic power* of \mathfrak{a} w.r.t. N , denoted by $(\mathfrak{a}N)^{(n)}$, is defined to be the intersection of those primary components of $\mathfrak{a}^n N$ which correspond to the minimal elements of $\text{Ass}_R N/\mathfrak{a}N$. Then the \mathfrak{a} -adic filtration $\{\mathfrak{a}^n N\}_{n \geq 0}$ and the \mathfrak{a} -symbolic filtration $\{(\mathfrak{a}N)^{(n)}\}_{n \geq 0}$ induce topologies on N which are called the \mathfrak{a} -adic topology and \mathfrak{a} -symbolic topology, respectively. These two topologies are said to be linearly equivalent if, there is an integer $k \geq 0$ such that $(\mathfrak{a}N)^{(n+k)} \subseteq \mathfrak{a}^n N$ for all integers $n \geq 0$.

Our main point of the present paper concerns an investigation of the linearly equivalent of the \mathfrak{a} -symbolic and the \mathfrak{a} -adic topology topologies on N . More precisely we shall show that:

Theorem 1.1. *Let R be a commutative Noetherian ring, and let N be a non-zero finitely generated R -module. Then for any N -proper ideal \mathfrak{a} of R , the \mathfrak{a} -symbolic topology on N is linearly equivalent to the \mathfrak{a} -adic topology on N if and only if, for every $\mathfrak{p} \in \text{Supp}(N)$, $\text{Ass}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ consists of a single prime ideal and $\dim N \leq 1$.*

The result in Theorem 1.1 is proved in Theorem 2.4. Our method is based on the theory of the asymptotic and essential primes of \mathfrak{a} w.r.t. N which were introduced by McAdam [6], and in [1], Ahn extended these concepts to a finitely generated R -module N . One of our tools for proving Theorem 1.1 is the following, which plays a key role in this paper.

Proposition 1.2. *Let R be a commutative Noetherian ring and \mathfrak{a} an ideal of R . Let N be a non-zero finitely generated R -module such that $\dim N > 0$, and let $\mathfrak{p} \in \text{Supp}(N) \cap V(\mathfrak{a})$. Then the following conditions are equivalent:*

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- (i) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$.
- (ii) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}\mathfrak{b}, N)$, for any N -proper ideal \mathfrak{b} of R with $\text{height}_N \mathfrak{b} > 0$.
- (iii) $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$, for any N -proper element x of R with $x \notin \bigcup_{\mathfrak{p} \in \text{mAss}_R N} \mathfrak{p}$.

A prime ideal \mathfrak{p} of R is called a *quitesessential* (resp. *quitasymptotic*) *prime ideal* of \mathfrak{a} w.r.t. N precisely when there exists $\mathfrak{q} \in \text{Ass}_{R_{\mathfrak{p}}}^* N_{\mathfrak{p}}^*$ (resp. $\mathfrak{q} \in \text{mAss}_{R_{\mathfrak{p}}}^* N_{\mathfrak{p}}^*$) such that $\text{Rad}(\mathfrak{a}R_{\mathfrak{p}}^* + \mathfrak{q}) = \mathfrak{p}R_{\mathfrak{p}}^*$. The set of quitesessential (resp. quitasymptotic) prime ideals of \mathfrak{a} w.r.t. N is denoted by $Q(\mathfrak{a}, N)$ (resp. $\bar{Q}^*(\mathfrak{a}, N)$) which is a finite set.

We denote by \mathcal{R} the *graded Rees ring* $R[u, \mathfrak{a}t] := \bigoplus_{n \in \mathbb{Z}} \mathfrak{a}^n t^n$ of R w.r.t. \mathfrak{a} , where t is an indeterminate and $u = t^{-1}$. Also, the *graded Rees module* $N[u, \mathfrak{a}t] := \bigoplus_{n \in \mathbb{Z}} \mathfrak{a}^n N$ over \mathcal{R} is denoted by \mathcal{N} , which is a finitely generated graded \mathcal{R} -module. Then we say that a prime ideal \mathfrak{p} of R is an *essential prime ideal* of \mathfrak{a} w.r.t. N , if $\mathfrak{p} = \mathfrak{q} \cap R$ for some $\mathfrak{q} \in Q(u\mathcal{R}, \mathcal{N})$. The set of essential prime ideals of \mathfrak{a} w.r.t. N will be denoted by $E(\mathfrak{a}, N)$.

Also, the *asymptotic prime ideals* of \mathfrak{a} w.r.t. N , denoted by $\hat{A}^*(\mathfrak{a}, N)$, is defined to be the set $\{\mathfrak{q} \cap R \mid \mathfrak{q} \in \bar{Q}^*(u\mathcal{R}, \mathcal{N})\}$.

In [14], Sharp et al. introduced the concept of integral closure of \mathfrak{a} relative to N , and they showed that this concept have properties which reflect some of those of the usual concept of integral closure introduced by Northcott and Rees in [12]. The integral closure of \mathfrak{a} relative to N is denoted by $\mathfrak{a}^{-(N)}$. In [11], it is shown that the sequence $\{\text{Ass}_R R/(\mathfrak{a}^n)^{-(N)}\}_{n \geq 1}$, of associated prime ideals, is increasing and ultimately constant; we denote its ultimate constant value by $\hat{A}^*(\mathfrak{a}, N)$. In the case $N = R$, $\hat{A}^*(\mathfrak{a}, N)$ is the asymptotic primes $\hat{A}^*(\mathfrak{a})$ of \mathfrak{a} introduced by Ratliff in [13]. Also, it is shown in [10, Proposition 3.2] that $\hat{A}^*(\mathfrak{a}, N) = \bar{A}^*(\mathfrak{a}, N)$.

If (R, \mathfrak{m}) is local, then R^* (resp. N^*) denotes the completion of R (resp. N) w.r.t. the \mathfrak{m} -adic topology. In particular, for every prime ideal \mathfrak{p} of R , we denote $R_{\mathfrak{p}}^*$ and $N_{\mathfrak{p}}^*$ the $\mathfrak{p}R_{\mathfrak{p}}$ -adic completion of $R_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$, respectively. For any ideal \mathfrak{b} of R , the *radical* of \mathfrak{b} , denoted by $\text{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$. Finally, for each R -module L , we denote by $\text{mAss}_R L$ the set of minimal prime ideals of $\text{Ass}_R L$.

Recall that an ideal \mathfrak{b} of R is called N -proper if $N/\mathfrak{b}N \neq 0$, and, when this the case, we define the N -height of \mathfrak{b} (written $\text{height}_N \mathfrak{b}$) to be

$$\inf\{\text{height}_N \mathfrak{p} : \mathfrak{p} \in \text{Supp } N \cap V(\mathfrak{b})\},$$

where $\text{height}_N \mathfrak{p}$ is defined to be the supremum of lengths of chains of prime ideals of $\text{Supp}(N)$ terminating with \mathfrak{p} . Also, we say that an element x of R is an N -proper element if $N/xN \neq 0$. For any unexplained notation and terminology we refer the reader to [3] or [7].

2. THE MAIN RESULT

Let R be a commutative Noetherian ring, and let N be a non-zero finitely generated R -module. The purpose of the present paper is to give an investigation of the linearly equivalent of the \mathfrak{a} -symbolic and the \mathfrak{a} -adic topology topologies on N . The main goal of this section is Theorem 2.4. The following proposition plays a key role in the proof of the main theorem.

Proposition 2.1. *Let \mathfrak{a} be an ideal of R and let N be a non-zero finitely generated R -module with $\dim N > 0$. Let $\mathfrak{p} \in \text{Supp}(N) \cap V(\mathfrak{a})$. Then the following conditions are equivalent:*

- (i) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$.
- (ii) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}\mathfrak{b}, N)$, for any N -proper ideal \mathfrak{b} of R with $\text{height}_N \mathfrak{b} > 0$.
- (iii) $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$, for any N -proper element x of R with $x \notin \bigcup_{\mathfrak{p} \in \text{mAss}_R N} \mathfrak{p}$.
- (iv) $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$, for some N -proper element x of R with $x \notin \bigcup_{\mathfrak{p} \in \text{mAss}_R N} \mathfrak{p}$.

Proof. (i) \implies (ii): Let $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$ and let \mathfrak{b} be an N -proper ideal of R such that $\text{height}_N \mathfrak{b} > 0$. Then, in view of [10, Remark 2.4],

$$\mathfrak{p}/\text{Ann}_R N \in \hat{A}^*(\mathfrak{a} + \text{Ann}_R N / \text{Ann}_R N).$$

Hence, as by [8, Theorem 2.1],

$$\text{height}_N \mathfrak{b} = \text{height}(\mathfrak{b} + \text{Ann}_R N / \text{Ann}_R N) > 0,$$

it follows from [5, Proposition 3.26] that

$$\mathfrak{p}/\text{Ann}_R N \in \hat{A}^*(\mathfrak{a}\mathfrak{b} + \text{Ann}_R N / \text{Ann}_R N).$$

Therefore by using [10, Remark 2.4], we obtain that $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}\mathfrak{b}, N)$, as required.

(ii) \implies (iii): Let (ii) hold and let x be an N -proper element of R such that $x \notin \bigcup_{\mathfrak{p} \in \text{mAss}_R N} \mathfrak{p}$. Then it is easy to see that $\text{height}_N xR > 0$, and so according to the assumption (ii), we have $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$.

(iii) \implies (iv): Since $\dim N > 0$, there exists $\mathfrak{q} \in \text{Supp } N$ such that $\text{height}_N \mathfrak{q} > 0$. Hence $\mathfrak{q} \not\subseteq \bigcup_{\mathfrak{p} \in \text{mAss}_R N} \mathfrak{p}$, and so there is $x \in \mathfrak{q}$ such that $x \notin \bigcup_{\mathfrak{p} \in \text{mAss}_R N} \mathfrak{p}$. Consequently, it follows from the hypothesis (iii) that $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$.

(iv) \implies (i): Let x be an N -proper element of R such that $x \notin \bigcup_{\mathfrak{p} \in \text{mAss}_R N} \mathfrak{p}$ and let $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$. Then

$$\mathfrak{p}/\text{Ann}_R N \in \hat{A}^*(x\mathfrak{a} + \text{Ann}_R N / \text{Ann}_R N),$$

by [10, Remark 2.4]. Now, since $x \notin \bigcup_{\mathfrak{p} \in \text{mAss}_R N} \mathfrak{p}$, it is easy to see that $x + \text{Ann}_R N$ is not in any minimal prime $R/\text{Ann}_R N$. Therefore, it follows from [5, Proposition 3.26] that

$$\mathfrak{p}/\text{Ann}_R N \in \hat{A}^*(\mathfrak{a} + \text{Ann}_R N / \text{Ann}_R N).$$

Consequently, in view of [10, Remark 2.4], $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$, and this completes the proof. \square

Before we state Theorem 2.4 which is our main result, we give a couple of lemmas that will be used in the proof of Theorem 2.4.

Lemma 2.2. *Let (R, \mathfrak{m}) be a local ring and let N be a non-zero finitely generated R -module such that $\dim N > 0$ and that $\text{Ass}_R N$ has at least two elements. Then there is an ideal \mathfrak{a} of R such that $\mathfrak{m} \in Q(\mathfrak{a}, N) \setminus \text{mAss } N/\mathfrak{a}N$.*

Proof. See [2, Proposition 4.2]. \square

Lemma 2.3. *Let N be a non-zero finitely generated R -module and let \mathfrak{a} be an N -proper ideal of R . Then $E(\mathfrak{a}, N) = \text{mAss}_R N/\mathfrak{a}N$ if and only if the \mathfrak{a} -symbolic topology is linearly equivalent to the \mathfrak{a} -adic topology.*

Proof. The assertion follows easily from [9, Theorem 4.1]. \square

We are now ready to state and prove the main theorem of this paper which is a characterization of the certain modules in terms of the linear equivalence of certain topologies induced by families of submodules of a finitely generated module N over a commutative Noetherian ring R . We denote by $Z_R(N)$ the set of zero divisors on N , i.e., $Z_R(N) := \{r \in R \mid rx = 0 \text{ for some } x(\neq 0) \in N\}$.

Theorem 2.4. *Let N be a non-zero finitely generated R -module. Then the following conditions are equivalent:*

(i) *For every N -proper ideal \mathfrak{b} of R , the \mathfrak{b} -symbolic topology is linearly equivalent to the \mathfrak{b} -adic topology.*

(ii) *$\dim N \leq 1$ and $\text{Ass}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ consists of a single prime ideal, for all $\mathfrak{p} \in \text{Supp}(N)$.*

Proof. Suppose that (i) holds. Firstly, we show that $\dim N \leq 1$. To achieve this, suppose the contrary is true. That is $\dim N > 1$. Then there exists $\mathfrak{p} \in \text{Supp}(N)$ such that $\text{height}_N \mathfrak{p} > 1$. Hence $\mathfrak{p} \not\subseteq \bigcup_{\mathfrak{q} \in \text{mAss}_R N} \mathfrak{q}$, and so there exists $x \in \mathfrak{p}$ such that $x \notin \bigcup_{\mathfrak{q} \in \text{mAss}_R N} \mathfrak{q}$. Now, since $\mathfrak{p} \in \bar{A}^*(\mathfrak{p}, N)$ and $xN \neq N$, it follows from Proposition 2.1 that $\mathfrak{p} \in \bar{A}^*(x\mathfrak{p}, N)$. Therefore, in view of [1, Theorem 3.17] we have $\mathfrak{p} \in E(x\mathfrak{p}, N)$.

On other hand, since $x \notin \bigcup_{\mathfrak{q} \in \text{mAss}_R N} \mathfrak{q}$, it is easily seen that $\mathfrak{p} \notin \text{mAss}_R N/x\mathfrak{p}N$, and so by the assumption (i) and Lemma 2.3 we have $\mathfrak{p} \notin E(x\mathfrak{p}, N)$, which is a contradiction. Hence, $\dim N \leq 1$. Now, we show that $\text{Ass}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ consists of a single prime ideal, for all $\mathfrak{p} \in \text{Supp}(N)$. To do this, if $\dim N = 0$, then $\dim N_{\mathfrak{p}} = 0$. Hence $\text{Ass}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} = \{\mathfrak{p}R_{\mathfrak{p}}\}$, as required. Consequently, we have $\dim N_{\mathfrak{p}} = 1$. Now, if $\text{Ass}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ has at least two elements, then in view of Lemma 2.2 there exists an ideal $\mathfrak{a}R_{\mathfrak{p}}$ of $R_{\mathfrak{p}}$ such that $\mathfrak{p}R_{\mathfrak{p}} \in Q(\mathfrak{a}R_{\mathfrak{p}}, N_{\mathfrak{p}})$ but $\mathfrak{p}R_{\mathfrak{p}} \notin \text{mAss}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}$. Therefore, in view of [1, Lemma 3.2 and Theorem 3.17], $\mathfrak{p} \in E(\mathfrak{a}, N) \setminus \text{mAss}_R N/\mathfrak{a}N$, which is a contradiction.

In order to show the implication (ii) \implies (i), in view of Lemma 2.3 it is enough for us to show that $E(\mathfrak{b}, N) = \text{mAss}_R N/\mathfrak{b}N$. To this end, let $\mathfrak{p} \in E(\mathfrak{b}, N)$. By virtue of [1, Lemma 3.2], we may assume that (R, \mathfrak{p}) is local.

Firstly, suppose $\dim N = 0$. Then it readily follows that $\mathfrak{p} \in \text{mAss}_R N/\mathfrak{b}N$, as required. So we may assume that $\dim N = 1$. There are two cases to consider:

Case 1. $\mathfrak{b} \not\subseteq Z_R(N)$. Then $\text{grade}(\mathfrak{b}, N) > 0$. Since $\dim N = 1$, it follows that $\text{height}_N \mathfrak{b} = 1$, and so $\mathfrak{b} + \text{Ann}_R N$ is \mathfrak{p} -primary. Hence $\mathfrak{p} \in \text{mAss}_R N/\mathfrak{b}N$, as required.

Case 2. Now, suppose that $\mathfrak{b} \subseteq Z_R(N)$. Then there exists $z \in \text{Ass}_R N$ such that $\mathfrak{b} \subseteq z$. Since $\text{Ass}_R N$ consists of a single prime ideal, so $\text{Ass}_R N = \{z\}$. Hence in view of [1, Proposition 3.6], $\mathfrak{p}/z \in E(\mathfrak{b} + z/z, R/z)$. Since $\mathfrak{b} \subseteq z$, it follows from [4, Remark 2.3] that $\mathfrak{p} = z$, which is a contradiction, because $\dim N = 1$. Consequently, $\mathfrak{b} \not\subseteq Z_R(N)$ and the claim holds. \square

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